Solution to Assignment 3

Section 6.3.

(10b) The function $x(1+3/x)$ can be expressed as $f(x)/g(x)$ where $f(x) = (1+3/x)$ and $g(x) = 1/x$. The limit $\lim_{x\to 0} x(1+3/x) = \lim_{x\to 0} f(x)/g(x)$ is of ∞/∞-type. We have

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{-3/x^2}{-1/x^2} = 3\ .$$

By L'Hospital Rule II,

$$\lim_{x\to 0} x(1+3/x) = 3.$$

Finally, by the continuity of the exponential function,

$$\lim_{x\to 0}\left(1+\frac{3}{x}\right)^x = \lim_{x\to 0} e^{x\log(1+3/x)} = e^{\lim_{x\to 0} x\log(1+3/x)} = e^3\ .$$

(11b) As the previous problem, consider its log expression first. Letting $f = \log\sin x$ and $g = 1/x$, we get

$$\lim_{x\to o^+}\frac{\log\sin x}{1/x} = \lim_{x\to 0^+}\frac{\cos x/\sin x}{-1/x^2} = 0\ ,$$

so

$$\lim_{x\to 0^+}(\sin x)^x = \lim_{x\to 0^+} e^{x\log\sin x} = e^{\lim_{x\to 0^+}(x\log\sin x)} = e^0 = 1\ .$$

(14) Routine. Use the formulas

$$(c^x)' = \log c\ c^x\ , \quad (x^x)' = (\log x + 1)x^x\ .$$

Section 6.4.

(9) By differentiating the sine function repeatedly we get

$$\frac{d^{2k}}{dx^{2k}}\sin x = (-1)^k\sin x, \quad \frac{d^{2k+1}}{dx^{2k+1}}\sin x = (-1)^k\cos x, \quad k\ge 0\ .$$

By Taylor's Expansion Theorem,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \frac{(-1)^n\cos c}{(2n+1)!}x^{2n+1}\ ,$$

or

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n\frac{x^{2n-1}}{(2n-1)!} + \frac{(-1)^n\sin c}{(2n)!}x^{2n}\ ,$$

for some mean value c. Using $|\sin c|, |\cos c| \le 1$, the remainder is bounded by

$$\frac{|x|^{2n+1}}{(2n+1)!}\ ,\ \text{or}\ \frac{|x|^{2n}}{(2n)!}\ ,$$

which tends to 0 as $n\to\infty$.

(10) The function $h(x) = e^{-1/x^2}$ when $x \neq 0$ and $h(0) = 0$. We claim that for each $k \geq 0$, there exists a polynomial $p_k(1/x)$ such that

$$h^{(k)}(x) = p_k(1/x)e^{-1/x^2} \ .$$

We use induction. Clearly this is true when $k = 0$ where $p_0 = 1$. Assuming it is true for n, we differentiate this relation (k replaced by n) to obtain

$$\begin{aligned} h^{(n+1)}(x) &= (h^{(n)})'(x) \\ &= (p_n(1/x))'e^{-1/x^2} + p_n(1/x)\frac{2}{x^3}e^{-1/x^2} \\ &= \left(-x^{-2}p_n'(1/x) + 2x^{-3}p_n(1/x)\right)e^{-1/x^2} \ . \end{aligned}$$

We can take $p_{n+1}(1/x) = -x^{-2}p_n'(1/x) + 2x^{-3}p_n(1/x)$. So far we have been assuming $x \neq 0$. Now, we claim that $h^{(k)}(0)$ exist and equal to 0 for all $k \geq 0$. Again we use induction. When $k = 0$ this follows from the definition of h. Assuming it is true for $k = n$, we have

$$\frac{h^{(n)}(x) - h^{(n)}(0)}{x - 0} = \frac{h^{(n)}(x)}{x - 0} = \frac{p_n(1/x)e^{-1/x^2}}{x} \ .$$

Letting $y = 1/x$, we have

$$\lim_{x \to 0^+} \frac{p_n(1/x)e^{-1/x^2}}{x} = \lim_{y \to \infty} \frac{yp_n(y)}{e^{y^2}} = 0 \ .$$

We conclude that the right derivative of $h^{(n)}$ exists and equal to 0 at $x = 0$. Similarly we can show the same result holds for the left derivative.

The interesting thing about this function is that it is smooth (infinitely many times differentiable) everywhere and all derivatives vanish at the origin. Its Taylor polynomials all vanish. Therefore, its Taylor Remainder is not small (as compared to its polynomial) at all in its Taylor expansion.

Supplementary Problems

1. Establish the following limits: For $\alpha > 0$,

   (a)
   $$\lim_{x \to \infty} \frac{x^\alpha}{e^x} = 0 \ .$$

   (b)
   $$\lim_{x \to \infty} \frac{\log x}{x^\alpha} = 0 \ .$$

   (c)
   $$\lim_{x \to 0^+} x^\alpha \log x = 0 \ .$$

   Solution. (b) and (c) follow from (a). We only prove (a). Using the expansion

   $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ ,$$

   and observing $x > 0$, we have

   $$e^x > \frac{x^n}{n!} \ ,$$

for every n. Given $a > 0$, we fixed a large $n_0, n_0 \geq a$. Using

$$e^x \geq \frac{x^{n_0+1}}{(n_0+1)!},$$

we have

$$0 \leq \frac{x^a}{e^x} \leq \frac{(n_0+1)!x^{n_0}}{x^{n_0+1}} = \frac{(n_0+1)!}{x} \ , \quad x \geq 1 \ .$$

Letting $x \to \infty$, (a) follows from Sandwich Rule.

2. Show that for $x \in (-1, 1]$,

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \ .$$

Solution. We have

$$\frac{d^k}{dx^k}\log(1+x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \ , \quad k \geq 1 \ .$$

Therefore, by Taylor's Expansion Theorem (taking $x_0 = 0$)

$$\log(1+x) = x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \frac{(-1)^n x^{n+1}}{(1+c)^n(n+1)} \ ,$$

where c is between 0 and x. When $x \in [0, 1], 1/(1+c) \leq 1$,

$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n}\right)\right| \leq \frac{1}{(n+1)} \to 0 \ ,$$

as $n \to \infty$ for $x \in [0, 1]$. On the other hand, when $x \in (-1, 0)$, $c \in (x, 0)$. Using $1/(1+c) \leq 1/(1+x)$,

$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \cdots + \frac{(-1)^{n-1}x^n}{n}\right)\right| \leq \frac{1}{(n+1)} \frac{x^{n+1}}{(1+x)^n} \ .$$

Therefore, when $|x| \leq (1+x)$, that $x \geq [-1/2, 1)$, $|x|^{n+1}/(1+x)^n \leq 1$ and

$$\lim_{n\to\infty} \frac{1}{(n+1)} \frac{x^{n+1}}{(1+x)^n} = 0 \ ,$$

and the conclusion follows.

Note. No conclusion is drawn when $x \in (-1, -1/2)$. Actually, the remainder still tends to 0 in this case but we need to use the integral form of the remainder which will be treated in a few weeks. When $x > 1$, the n-th term in its Taylor series, x^n/n, tends to ∞ as $n \to \infty$. Hence it is not convergent. We conclude that the Taylor series of $\log(1+x)$ converges to the function itself if and only if $x \in (-1, 1]$.

3. Let

$$q(x) = -12 + x^2 + 3x^4.$$

Determine the coefficients in

$$q(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3 + a_4(x-1)^4 \ .$$

Solution. Use Taylor's Expansion Theorem at $x = 1$. We have $q'(x) = 2x + 12x^3$, $q''(x) = 2 + 36x^2$, $q'''(x) = 72x$, and $q^{(4)}(x) = 72$. Therefore,

$$\begin{aligned} q(x) &= q(1) + q'(1)(x-1) + q''(1)(x-1)^2/2 + q'''(1)(x-1)^3/6 + q^{(4)}(1)(x-1)^4/24 \\ &= -8 + 14(x-1) + 19(x-1)^2 + 12(x-1)^3 + 3(x-1)^4 \ . \end{aligned}$$

4. Let f be infinitely differentiable function. Suppose that there is a polynomial p of degree n such that for some $\delta, C > 0$,

$$|f(x) - p(x)| \le C|x - x_0|^{n+1} \ , \forall x \in [x_0 - \delta, x_0 + \delta] \ .$$

Show that p must be the n-th Taylor polynomial of f at x_0.

Solution. We first claim that if a polynomial q satisfies

$$|q(x)| = |b_0 + b_1(x - x_0) + b_2(x - x_0)^2 + \cdots + b_n(x - x_0)^n| \le C|x - x_0|^{n+1} \ .$$

Then $q \equiv 0$. This is obvious. First, setting $x = x_0$ in the inequality we get $b_0 = 0$. Next, divide both sides of this inequality by $(x - x_0)$ and then set $x = x_0$ we get $b_1 = 0$. Keep dividing the inequality by powers of $x - x_0$ and then letting $x = x_0$ yield $b_j = 0$ for all j.

Now, by Taylor's Expansion Theorem we have

$$f(x) = f(x_0) + \cdots \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R,$$

where the remainder R satisfies

$$|R| \le C_1|x - x_0|^{n+1} \ , \quad C_1 = \frac{1}{(n+1)!} \sup_{[x_0-1, x_0+1]} |f^{(n+1)}(x)| \ .$$

Let

$$p(x) = a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n \ .$$

We have

$$\left| (f(x_0) - a_0) + \left(\frac{f'(x_0)}{1!} - a_1 \right)(x - x_0) + \cdots + \left(\frac{f^{(n)}(x_0)}{n!} - a_n \right)(x - x_0)^n \right| \le (C + C_1)|x - x_0|^{n+1} \ .$$

By our claim we conclude that

$$a_j = \frac{f^{(j)}(x_0)}{j!} \ , \quad j = 0, 1, \cdots, n \ .$$